Module 6: Congruent and Similar Triangles

Topic 6 Content: Using Direct Proofs to Prove Triangles Similar Transcript

Hi, guys. Welcome to Geometry. This topic is going to focus on how to use direct proofs to prove triangles similar. Everything you know about similar triangles is going to come in handy during this lesson. You ready to get started? Let's go.

Before we dive into those direct proofs, let's just review the three ways that we know of how to prove that triangles are similar. Let's get these little boxes out of our way here. Side-Side-Side Similarity. If you remember, if you're given a pair of triangles, if you can prove that you have three pairs of corresponding sides proportional, then you can prove that you have a pair of similar triangles.

Let's take a look at the next one, Side-Angle-Side Similarity. In this case, if you have a pair of triangles and you can prove that you have two pairs of corresponding sides proportional and one pair of included angles congruent, then you can conclude that you have a pair of similar triangles.

The last way, Angle-Angle Similarity. In this case, you have your pair of triangles. You don't actually have to worry about proving any side relationships at all. For this case, if you have two pair of corresponding angles congruent, then you can prove that you have a pair of similar triangles. Let's keep these three ways in mind as we start diving into these direct proofs.

I want to start here by actually talking a little bit about a relationship inside of a right triangle. Given a right triangle, if we sketch an altitude, what an altitude is, it's a segment that has its endpoints. One is on the vertex and the other endpoint of the segment is on the side opposite that vertex and that segment is perpendicular to that opposite side. That's a lot of things that I just said with that one, but that is how we describe an altitude. It's a segment drawn from a vertex to the side opposite the vertex and that altitude is perpendicular to that opposite side. When you're given a right triangle, if you sketch an altitude, you actually create three right triangles. Well, you create two and then you have the one that you are given. There are some relationships and special relationships between those triangles.

Let's take a look here. I've separated our group of triangles. I've written them out, sketched them out individually so that we can pull apart and really dig into those relationships. Here we have our altitude in gray and let's look at what we have here. I've changed the orientation of these three triangles so that each triangle is lying on its hypotenuse and the right angle is up at the top. What we have here, triangle ACD and that's our triangle here on the left. Again, I've changed the orientation so the triangle is lying on its hypotenuse. Then we have triangle ABC which is actually the large triangle that we started with. Then we have triangle CBD and that's this triangle here on the right. Again, I have it lying on its hypotenuse. We started out with one triangle. We sketched that altitude and actually ended up with three triangles at the end and there's a special relationship that exists between those triangles.

Let's first start out by looking at triangle ACD and triangle ABC. I see here that each triangle has a right angle. These are right triangles. I know that angle D is congruent to angle C because all right angles are congruent. They all measure 90 degrees. Then I see here that each of these triangles also includes angle A. Angle A has to be congruent to itself by the reflexive property. I actually have in these two triangles angle A congruent to angle A by reflexive property and angle D congruent to angle C because all right angles are congruent. What that means is that with these two triangles I could say; let's switch the blocking here, triangle ACD is similar to triangle ABC. I know that because of the Angle-Angle Similarity relationship that exists between these two triangles. I have two pairs of corresponding angles congruent. Keep that in mind. Going to keep that similarity statement there. I'm going to erase this.

Let's switch and let's focus on these two triangles. I see here I have angle C congruent to angle D because all right angles are congruent. If I look, each of these triangles includes angle B. I could say that angle B is congruent to angle B again by the reflexive property. Looking at these two triangles I see that I actually have two pair of corresponding angles that are congruent. Again, I can say that triangle ABC is similar to triangle CBD and again that's because of Angle-Angle Similarity. We have our two pairs of corresponding angles that are congruent. Stay with me. We're going to get to ... We really wanted to get to here.

Let's take a step back and really look what happened here. We proved that that triangle ACD was similar to triangle ABC. We proved that triangle ABC is similar to triangle CBD. What we finally can conclude is this; let's write this final similarity statement. We can say that triangle ACD is similar to triangle CBD. We started out by proving a relationship between these two triangles the first two, then we proved the relationship between these two triangles the second two, and then allowed us to establish a relationship between our first and our last triangle. We actually can justify that relationship by the transitive property.

If you remember the transitive property, it's like the chain rule that links together a lot of different relationships. If we look here—I'm going to block some things off in color—here's the triangle ACD similar to triangle ABC and triangle ABC was similar to triangle CBD. Triangle ABC was that link that allowed us to establish a relationship between triangle ACD and triangle CBD. We can say that those triangles are similar by the transitive property. I just wanted you to see that because that is the special relationship that exists when you're dealing with right triangles. You have a right triangle and you sketch an altitude, you actually create three pairs of similar triangles. Let's keep going with this.

Let's take a look at this. We're going to use coordinate methods to prove that triangle VUT is similar to triangle MNL by Side-Side-Side Similarity. Look here, I'm on the coordinate plane. Like I said, I have to use coordinate methods. I'm asked to establish the similarity relationship by side-Side-Side Similarity. What that means is I need to prove that I have three pairs of corresponding sides that are proportional. Let's start digging into this coordinate plane and get some side lengths that we can use here.

Let's start with the triangle on the left. UV is two units. It's a horizontal length, so I can just count it. TU, let's see 1, 2, 3 that's 4 units. TV, that's a diagonal length, so I can't count it like I did the other sides. I'm going to need to use the distance formula to figure out the length of TV. I'm going to scroll back and forth here a little bit, scroll up and down. To use the distance formula I'm going to come, actually I'm going to come right underneath here and write my distance formula.

The distance formula is the square root of x_2 minus x_1 squared plus y_2 minus y_1 squared. It's the square root of that quantity in there. I need to get the ordered pairs for T and V. T that's negative 1, 2, 3, 4, 5, and 2, those are the coordinates for T, and for V −1, 2, 3, −2. I label one of these x_1, y_1, the other x_2, y_2 and then let's go ahead and use the distance formula so we can determine the length of TV. I'm going to have to scroll up and down here, so let's see.

The first thing I need is x_2 minus x_1, so that is −3 minus −5; −3 minus −5 squared. Let's get a little more workspace here. Let's extend this page. Then I need y_2 minus y_1, so that is −2 minus 2 squared. Let's simplify this expression here. The first thing jumping out it means this double negative in here and that's going to become a plus. Let's start this simplified. −3 plus 5, that's 2, so 2 squared. −2 minus 2, that's −4, so −4 squared. Keep working here. 2 squared, that's 4, −4 squared, that's 16, so 4 plus 16, that's 20, so the square root of 20.

Because 20 isn't a perfect square, I'm just going to leave it exact so that I don't have to round because it actually may affect my answer on the end if I have to round this value. I'm just going to leave it here and say TV is the square root of 20 units long.

For the other triangle, actually I want you to do this work over here. I want you to determine the length of MN, of LN, and of LM. Before we get too crowded in here, that's what we said before, the math starts to get crowded, it can start to get confusing so let's get rid of a few things here. We're going to keep that distance formula up there for you, but I'm just going to erase the work that went along with that first triangle. There we go. I'm going to pause. I want you to take a few minutes and work your way through getting the lengths of each side of this triangle. Press play when you're ready to check your answers.

Let's see what you got. For MN, let me make sure I have my pen. MN, that's 1 unit long. LN, that's 2 units long. LM is a diagonal length, so you're going to have to use the distance formula there, so let's get the coordinates of those endpoints. L is at (3, 0). M is at (2, −2). We'll label (x_1, y_1) (x_2, y_2), and then let's go ahead and start using that distance formula. I'm going to start with x_2 minus x_1, so that's 2 minus 3, so 2 minus 3 squared. y_2 minus y_1, so that's −2 minus 0, so −2 minus 0 squared.

Let's simplify here. 2 minus 3, that's −1, so −1 squared. −2 minus 0, that's −2, so −2 squared. Keep simplifying and that when you're working with the distance formula. Unless you start to skip steps, if you get more comfortable, it does have a lot of steps. −1 squared, that's 1. −2 squared, that's 4. 1 plus 4, that's 5. I end up with the square root of 5. Because 5 is not a perfect square, I'm not going to approximate its value. I'm just going to leave it as an exact value and say that LM is the square root of 5 units long.

Now that we have the lengths of each side of these triangles, we need to determine if we actually have three pairs of corresponding sides proportional. If you recall from your work with similar triangles before, in order to determine if you have that relationship, you're going to need to check those ratios. Let me show you what I mean. Let's get some of this work out of our way here. We've gotten what we needed from the distance formula. Let's erase this and get some workspace. Let's get rid of that. Here we go.

Let's see what we have here. UV is the smallest side in my triangle on the left and MN is the smallest side of my triangle on the right. They increase in value. I wrote it like that because I knew we were going to get to this step. These side lengths go from smallest to largest on each triangle. Let's test out these ratios and see if we have three pairs of corresponding sides proportional. If we do, then what that will mean is that the relationship of UV to MN that ratio will be equal to TU to LN and then finally TV to LM. Each of these ratios should be congruent if I'd actually have three pairs of corresponding sides that are proportional. The relationship of UV to MN should be the same as the relationship of TU to LN and of TV to LM. Let's test that out.

UV to MN, let's get that. UV is 2 units. MN is 1 unit. The relationship is 2 to 1. Let's check out the second pair TU to LN, so TU to LN. I'm going to scroll again. TU is 4. LN is 2. That is 4 to 2, which you can simplify to represent as 2 to 1. So far those relationships are the same.

Then let's check out that last pair TV to LM. Let's see here, TV to LM. TV is the square root of 20 and LM is the square root of 5, so the square root of 20 over the square root of 5.

We have to go back and use some of those algebra skills because we're dealing with radicals here. Square root of 20 divided by the square root of 5, I can simplify by just really focusing on the value underneath the radical. I can represent this as the square root of 4 because 20 divided by 5 is 4. The square root of 20 divided by the square root of 5 is the square root of 4. Simplify this even further and you know that the square root of 4 is 2. Then recall that you can represent any whole number as a fraction by just representing it with denominator of 1.

If I look back and compare each of these ratios, each of those relationships was a 2 to 1 relationship, each pair 2 to 1. When we were trying to determine if we had three pairs of corresponding sides proportional, we do because each of those ratios is the same. We've shown here is some algebraic methods and some coordinates methods that we do actually have a pair of similar triangles. Taking you back to what we actually were asked to prove, we've proven that triangle VUT is similar to triangle MNL by Side-Side-Side Similarity. If you wanted to write that out formally, you could say that you have three pairs of corresponding sides. I'm going to abbreviate that, proportional. Good job on that. That tied in a lot of algebra toward geometry here, so good work on that one.

Let's take a look at this two-column proof. We're given that segment ED is parallel to segment AC and we're asked to prove that triangle BED is similar to triangle BAC, if you recall from our work before with two-column proofs, you're given a set of statements and then you have to justify them with a set of reasons. We're going to use all the facts and relationships and definitions that we know to be true in order to justify each of these statements that we have on the left. Our two-column proof starts out with segment ED is parallel to AC. Like most two-column proofs, it begins with a given statement. The reason that I know the statement is true; start my pen, is because it was given information. It was given to us at the beginning of the problem.

I'm going to go ahead and mark this relationship on my triangles. You see here we have triangle BED and triangle BAC. We have a triangle inscribed, or sitting inside another triangle. I'm going to separate those also just so you have an idea of what triangles look like together as well as what they look like separately. Sometimes it's easier to justify those relationships when you look at the triangles apart instead of one side of another. I'm going to sketch triangle BED. Let's get some more space here. I'm going to scroll again.

BED is that small triangle at the top and then we have BAC which is our larger triangle. We are marking that initial relationship we were given that ED is parallel to AC. We're going to mark—remember that's the symbol for parallel. Again, let's mark it on our triangles that are separated down here. Here we go. Go back again. We're scrolling a lot here. Let's get to that second statement.

We're told that angle BED is congruent to angle BAC. It may not be clear at first why those angles are congruent. I think it will help you if you mark where those angles are in the figure. It might really help you see what that relationship is or what that reason is that justifies that relationship. We have angle BED; I'm going to switch colors here putting in red, congruent to angle BAC. If you still don't see it; let me show you something here. I think I'll stick to red. When we were told that ED is parallel to AC, what that means is if I extend those lines and extend this side of the triangle; I'm going to group those together and I'm going to show you something here. Let's get the triangle to say. I want that to sit there.

Let's lock that in place, but let's get those other marks. Let's group those together and let's slide them down.

Do you recall from your earlier geometry or what we've covered before what this situation relates to? If you just look at it here, you have a pair of parallel lines intersected by a transversal. Right where we marked those angles that were congruent, those are actually a pair of corresponding angles. Like you know if you have a transversal intersecting a pair of parallel lines, your corresponding angles are congruent. That's the relationship that allows us—or that's the reason that we're able—to justify that relationship. Angle BED is congruent to angle BAC because they also are pair of corresponding angles. Like we were told before, we have two parallel lines intersected by a transversal if you look at it in that way, so you know that the corresponding angles have to be congruent.

Let's get that over here in our proof. We're going to put that right up here just so you can even lie that right back on top so can really see that again that those are pair of corresponding angles there that are congruent. Let's get that reason in our column. I'm going to abbreviate it because it is a little lengthy. We have a transversal intersecting parallel line and the corresponding angles are congruent. We have that in our picture here. Let's also get it in our triangles that are separated, get that angle relationship in there. Let's get back to the column up here.

We're also told that angle B is congruent to angle B. Remember: Anytime you have something that's congruent to itself, you can justify that by the reflexive property. Let's get that reason in there, so reflexive property. That's how we know that angle B is congruent to angle B. I'm going to get that in our figure here. Let's actually clean this up a little bit. We can get rid of that diagram that allowed us to justify those angle relationships. Let's just get this here, put the parallel symbol back. There we go.

Right here I've marked that angle B is congruent to itself. I used two arcs because I had already used one arc to show that those angles were congruent. I'm going to go back down here and get the same mark in my triangles that are separated. We can see here the last line of our proof, triangle BED is similar to triangle BAC. The last line of our proof is establishing that similarity statement so that means that we need to finish it off with either Side-Side-Side Similarity, Side-Angle-Side Similarity, or Angle-Angle Similarity.

If we look back at how we've marked our triangle, we have two pairs of corresponding angles that are congruent. The reason that I know these triangles are similar is by Angle-Angle Similarity. If we look back, we've taken care of every line of this two-column proof. We started with our given information, and that we were asked to prove by giving a set of justifications for each of these statements. Good job on that.

We've reached the point that it's your turn. I'm going to press pause, or I'm going to pause and take a few minutes and I want you to work your way through this example. When you're ready to check your answer, go ahead and press play.

Let's see how you did here. You were asked to prove that triangle TPQ is similar to triangle RSQ. I am given some side measures here. I'm starting to get a feel for how I may have to go about proving these triangles are similar. What I mean by that is I'm only given two side lengths on each triangle. I know that I can't use Side-Side-Side Similarity to prove that these triangles are similar because I don't have a measure for this third side. It's either going to be Angle-Angle Similarity or Side-Angle-Side Similarity. Because I was given side lengths, I think it's going to be Side-Angle-Side Similarity. I'm going to go that route.

Let's start trying to establish those side relationships between these two triangles. If you recall from our earlier practice with similar triangles, we would mark our smallest side, largest side. Then over here the smallest side, the largest side. Just to keep straight when we're trying to match up the side in one triangle that corresponds to the side in the other triangle, if you mark its relationship to the other side length from the given triangle that this is the smallest side in this triangle, it will correspond to the smallest side in the other triangle and then the same for those larger sides.

What we're going to try to prove here is that TQ, the relationship of TQ to QR to the smallest side in one to the smallest side in the other is the same as the relationship of PQ to

QS. If we can prove that, then we've shown that we have two pairs of corresponding sides that are proportional.

Let's check out the math here and see if it's actually true. TQ, I'm going to do the work down here, so TQ to QR. TQ is 3 units. QR is 12 units. If I simplify that, that will simplify 1 to 4. I know that the relationship of TQ to QR is 1 to 4. Let's check out PQ to QS. Get a little more workspace here, so PQ to QS. Scroll a little bit here. PQ is 7 and QS is 28, so 7 to 28. That will also reduce, that will also simplify to 1 to 4. Again, back up to the relationship up here, PQ to QS, 1 to 4. Those ratios were equal. We've shown here that we have two pairs of corresponding sides proportional.

We've tackled the sides. Let's try to get an angle relationship. A rule of thumb is anytime you have, this is sometimes referred to as the bow-tie problem or the bow-tie image because this resembles a bow-tie if you look at this figure whole, like it's one whole image instead of two separate triangles. Anytime that you have the bow-tie problem as referred to it, you always have a relationship or you typically have a relationship where you have vertical angles right in the center.

If you look here—I'm going to erase these marks here just you can focus your attention on the angle relationships—you have a pair of vertical angles in the center. Those vertical angles are congruent because that's what we know about vertical angles. Anytime you have the bow-tie problem, which really is a way of saying a pair of intersecting lines, you're going to have some vertical angles that are congruent.

Here, if we wrote that angle relationship—let's get rid of the work that we did here for the sides and put it right here—you could say that angle PQT is congruent to angle SQR because vertical angles are congruent. What that shows us here is along with my two pairs of corresponding sides that are proportional, I also have one pair of included angles; let's make sure that actually looks like the word *pair*, one pair of included angles that are congruent. I've taken care of what I need in order to show that these triangles are similar by Side-Angle-Side Similarity. We used some algebraic methods now that we know about angle relationships to prove that these triangles are similar by Side-Angle-Side Similarity. Good job on that.

Guys, we've reached the conclusion of this topic on how to use direct proofs to prove that triangles are similar. I hope you saw how your knowledge of coordinate methods, everything you knew about similar triangles all came together for you in this lesson. Bye.